LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics

# Calculus IV

## Final Exam Spring 2010 (June 9, 2010)

Name: Solution ID:

| Question Number | Grade |
|---|---|
| 1. 4% | |
| 2. 6% | |
| 3. 6% | |
| 4. 5% | |
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| 8. 8% | |
| 9. 6% | |
| 10. 6% | |
| 11. 5% | |
| 12. 6% | |
| 13. 6% | |
| 14. 6% | |
| 15. 6% | |
| 16. 4% | |
| 17. 10% | |
| Tota 101 | |

1. Find the direction in which the function $f(x, y) = xy^2 - yx^2$ is increasing or decreasing most rapidly at the point $(-1, 2)$.

In the dir. of $\vec{\nabla} f = (y^2 - 2xy)\vec{i} + (2xy - x^2)\vec{j}$

(4%)

$\vec{\nabla} f = 8\vec{i} + 5\vec{j}$

Dir $= \frac{8i+5j}{\sqrt{89}}$ Most inc. ↗ ; and Most ↘ in dir. of $\frac{-8i+5j}{\sqrt{89}}$

2. Approximate the value of $9(4.01)^2(-7.99)^3$ using linearization.

$f(x,y) = 9x^2 - y^3$ @ $(4, -8)$ $f(4,-8) = 7344$

(1%)

$f_x = 18x = 72$ $f_y = -3y^2 = -192$

Value $\approx f(4,-8) + f_x(x-4) + f_y(y+8)$
$7344 + 72(0.01) - 192(0.01)$

3. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2$ on the curve $x = 8t + 1; y = 8t - 1, 0 \le t \le 1$.

(1%)

Two ways; we can think of $f(x,y)$ as

$f(t) = (8t+1)^2 + (8t-1)^2 = 64t^2 + 16t + 1 + 64t^2 - 16t + 1$
$= 132t^2 + 2$

$\Rightarrow f'(t) = 2(132)(t) \Rightarrow$ min or max @ $t=0$. $t=0$

Or $f_x = 2x = 0$ $f_y = 2y = 0 \Rightarrow$ Crit. pt @ $(0,0)$

$f_{xx} = f_{yy} = 2$

$f_{xy} = f_{yx} = 0$

$\Delta = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \Rightarrow$ @ $(0,0)$ same we have a minimum since $f_{xx} > 0$

then on the curve, use the param. as in the first method:

Conclusion: min. @ $(0, b)$ min $= 2$

4. Find parametric equations for the normal line to the surface $z = 7x^2 - 5y^2$ @ the point $(2, 1, 23)$

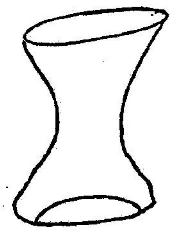
5%

$\vec{\nabla f} = 14x\,i - 10y\,j - \vec{k}\,\Big|_{(2,1,23)} = 28i - 10j - k.$

$$\begin{cases} x = 2 + 28t \\ y = 1 - 10t \\ z = 23 - t \end{cases}$$

5. Identify the type of surface represented by $\frac{x^2}{8} + \frac{y^2}{4} - \frac{z^2}{6} = 1$

hyperboloid.



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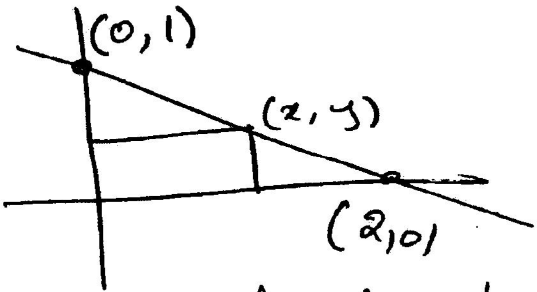
Since $(S) \cap (z=c)$ = ellipse

$(S) \cap \begin{pmatrix} x = a \\ y = c \end{pmatrix}$ : hyp.

Not passing through origin. All kinds of sym.

6. A rectangle with sides parallel to the axes is inscribed in the region bounded by the axes and the line $x + 2y = 2$. Find the maximum area of this rectangle.

5%




Maximize $f(x,y) = xy$ subject to constraint $g(x,y) = x + 2y = 2$

$\vec{\nabla f} = \lambda \vec{\nabla g}$

$yi + xj = \lambda(i + 2j)$

$$\begin{cases} y = \lambda \\ x = 2\lambda \\ x + 2y = 2 \\ 2\lambda + 2\lambda = 2 \end{cases} \Rightarrow (1, 1/2)$$

$\lambda = 1/2$

$x = 1$

Max @ $(1, 1/2)$

Max Area $= \frac{1}{2}$

6%

7. Find the volume of the region bounded by the coordinate planes, the parabolic cylinder $z = 49 - x^2$, and the plane $y = 4$

$$\int_0^7 \int_0^4 \int_0^{49-x^2} dz\, dy\, dx.$$

8%

8. Evaluate the integral $\iint_R (y-4x)(6x+y)\,dx\,dy$ where $R$ is the parallelogram bounded by the lines $y = 4x+8, y = 4x+10, y = -6x+4, y = -6x+7$, using a transformation.

Use $u = y - 4x$, $v = 6x + y$ $\qquad x = -\frac{u+v}{10}$, $y = \frac{6u+4v}{10}$

$$J = \begin{vmatrix} -1/10 & 1/10 \\ 6/10 & 4/10 \end{vmatrix} \Rightarrow |J| = \frac{1}{10}$$

G: $4 < v < 7$
$8 < u < 10$

$$\text{Int} = \int_4^7 \int_8^{10} uv\,|J|\,du\,dv = \frac{1}{10}\int_4^7 \int_8^{10} uv\,du\,dv$$

6%

9. Calculate the flow in the field $\vec{F} = (x-y)\vec{i} - (x^2+y^2)\vec{j}$ along the path $C$ from $(3,0)$ to $(-3,0)$ on the upper half of the circle $r = 3$.

$$\int_C F\cdot T\,ds = \iint N_x - M_y\,dA$$

Using Green's Theorem:

$$\int_0^{\pi} \int_0^3 (-2x-1)(r\,dr\,d\theta) \quad \text{use } x = r\cos\theta.$$

61.

10. Evaluate the work done by the force $\vec{F} = (y+z)\vec{i} + x\vec{j} + x\vec{k}$ over the segment between the points $(0,0,0)$ to $(7,4,7)$.First verify taht $F$ is conservative and use the most efficient method only.

Is $\vec{F}$ conservative? $M_y = N_x = 1$, $M_z = P_x = 1$, $N_z = P_y = 0$ ) yes ⇒

The potential $f$ of $\vec{F}$ is:

$f = \int f_x\,dx = \int y+z\,dx = yx + zx + h(y,z)$

$f_y = x + h_y(y,z) = N = x \Rightarrow h_y(y,z) = 0 \Rightarrow h(y,z) = g(z)$

$\therefore f = yx + zx + g(z)$

$f_z = x + g'(z) = N = x \Rightarrow g'(z) = 0 \Rightarrow g(z) = c$

$\therefore \boxed{f = yx + zx + c}$

$\therefore$ Answer $= f(7,4,7) - f(0,0,0) = 77$

%.

11. Find the outward flux of $\vec{F} = (-2x+9y)\vec{i} + (8x-9y)\vec{j}$ across the boundary of the region between $y = -3x^2 + 175$ and $y = 4x^2$ in the first quadrant.

Using Green's theorem

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R M_x + N_y\,dA$$

$$= \boxed{\int_0^5 \int_{4x^2}^{-3x^2+175} -2-9\,dy\,dx}$$

%.

12. Evaluate the surface integral of the function $g(x,y,z) = \dfrac{y}{\sqrt{36y^2+1}}$ over the surface of the parabolic cylinder $27y^2 + 9z = 9$ bounded by the planes $x = 0, x = 1, y = 0, z = 0$

$\iint_S g\,d\sigma =$

13. Find the outward flux of $\vec{F} = xy\,\vec{i} + y^2\,\vec{j} - 2yz\,\vec{k}$ across the boundary of the solid wedge cut from the first quadrant by the plane $y + z = 5$ and the cylinder $x = 9 - 25y^2$

6%

Div. theorem

Outward Flux $= \iiint_E \operatorname{div} F \, dV = \iiint M_x + N_y + P_z \, dV$

$= \iiint y + 2y - 2y \, dV$

$= \int_{3/5}^{5} \int_0^{(5-y)} \int_0^{(9-25y^2)} y \, dx\, dz\, dy$

14. Find the curvature of the space curve: $\vec{r}(t) = -3\vec{i} + (3+2t)\vec{j} + (t^2+7)\vec{k}$

6%

$K = \left|\frac{dT}{ds}\right| = \frac{|v \times a|}{|v|^3}$

$v = 2j + 2t\,\vec{k}$

$a = 2\vec{k}$

$v \times a = \begin{vmatrix} i & j & k \\ 0 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix}$

$= 4i + 4\vec{k}$

$|v| = \sqrt{4 + 4t^2} = 2\sqrt{1+t^2}$

$|v \times a| = \sqrt{16+16} = 4\sqrt{2}$

$= \frac{4\sqrt{2}}{2^3(1+t^2)^{3/2}}$

$K = \frac{1}{\sqrt{2}(1+t^2)^{3/2}}$

15. Find the torsion of the space curve $\vec{r}(t) = (7 + 9\sin(4t/9))\vec{i} + (3t)\vec{j} + (4 + 9\cos(4t/9))\vec{k}$

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$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$

(4%)

16. Write in your own words why the Fourier series are useful. Compare them to other important series you have learned.

They help us approximate periodic functions as trig. series.

They look like power series that approximate functions in Taylor's Theorem.

17. Consider the function $f(x) = \begin{cases} -8 & -\pi < x < 0 \\ 8 & 0 < x < \pi \end{cases}$

(10%)

(a) Find its Fourier series

Since $f$ is odd $\Rightarrow$ $a_n = 0$, $b_n \neq 0$. $T = 2\pi$

$$b_n = \frac{2 \cdot 2}{T} \int_0^{T/2} f(x) \sin(nx)\, dx$$

(8)

$$= \frac{2}{\pi} \int_0^{\pi} 8 \sin(nx)\, dx = \frac{-16}{\pi n} \cos nx \Big|_0^{\pi}$$

$$= -\frac{16}{\pi n} (\cos n\pi - 1) = \begin{cases} 0 & n: \text{even} \\ \frac{32}{\pi n} & n: \text{odd} \end{cases}$$

$\therefore$ Fourier Series:

$$= b_1 \sin \pi x + b_3 \sin 3\pi x + b_5 \sin 5\pi x + \cdots$$

$$* \quad = \frac{32}{\pi} \sin \pi x + \frac{32}{3\pi} \sin 3\pi x + \frac{32}{5\pi} \sin 5\pi x + \cdots$$

(2)

(b) Discuss convergence of that series.

$* \longrightarrow f(x)$ for $x \neq 0$

and

$* \longrightarrow 0$ @ $x = 0$.